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Properties of Physical Systems and Projectors: Dispersion Relations, Spectral Representations, Equations of Motion

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All physical systems should be described, as had been marked by von Neumann, via quantitative parameters and qualitative properties, such as causality, mass-spectrality, strict interval of work and so on, which are completely present or are absent, i.e. are representable by projectors. The description of such properties by equations of restriction for response functions, via projectors of the restricted domain (support) of parameters, is offered. Their Fourier transformations directly lead to spectral representations and dispersion relations (DRs) without preliminary study of analiticity. The offered method explaines their accordance to equations of motion. Among other examples the relativistic DRs are established and some their features are studied including possibilities and limits of instantaneous transfer of excitations. The DR for particle scattering on a fixed point-type scatterer is established via kinematics only, without definition of interactions type; it leads to determination of the unique numerical parameter, which can be suggested as a candidate on the role of universal bare charge of SUSY, etc.

KEY WORDS: property; dispersion relation; superluminal; bare charge.

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1. INTRODUCTION

According to the immortal "Mathematical Foundation of Quantum Mechanics" of John von Neumann (1932, Ch. 3.5) the complete description of physical systems must contain:(a) a set of quantitative characterizations (energy, positions, velocities, charges, etc.) and (b) a set of properties of states[2] (causality, restriction on the spectra of self energies, existence or absence of certain strictly isolated energy bands, strict combination of some quantative characteristics, etc.)

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[2] In the German original text "Eigenschaft," in the Russian translation "alternative property" that seems closer to the initial term. Its more general philosophical sense we do not consider, see e.g. Stanford Encyclopedia of Philosophy http://www.seop.leeds.ac.uk/archives/fall2000/entries/properties/

Formal relations dictated by the general principles of science (causality, laws of conservations, principles of thermodynamics) are of especial importance, including their prognostic role, at development of any theory. In the set of these relations are the Kramers-Kronig dispersion relations (DRs) and their generalizations (the general introduction and an initial history of DRs (Toll, 1956), their generalizations and applications in the quantum field theory (Bogoliubov and Shirkov, 1982), in the nonrelativistic physics (Nussenzweig, 1972)). But this developing had been based on investigations of analytic peculiarities of examined magnitudes that are not directly connected with properties of system in the von Neumann sense, and therefore they could obscure the physical context of these properties. As the analyticity was simply established at such way for energy only, this approach became very complicating even for one-particle functions with taking into account its momentum, in the relativistic theory or in the theory of space dispersion.

For the return to the more logical, as it seems, way we developed the direct use of the von Neumann statement for one particle relativistic functions (Perel'man, 1966). This approach was generalized on many-point functions of the axiomatic field theory (Perel'man, 1969) and on nonlinear systems (Perel'man, 1971). Some mathematical problems were considered in Perel'man (1976). As the relativistic DRs are very complicated and even intricated, some their approximated forms are given in Perel'man and Englman (2000).

Such DRs, despite of their mathematical complexity, are crucial and their role become decisive at consideration of some basic problems of the theory. So, in Perel'man (2005) on the base of the general theory of DRs, i.e. due the von Neumann conjecture, the possibilities of instantaneous transfer of excitations in the limits of near field and at very strict requirements to the energy spectra were established. These results, had completely described the recent observations of "superluminal" phenomena (e.g. reviews (Chiao and Steinberg, 1998; Milonni, 2002; Nimtz, 1997; Nimtz and Heitman, 1997; Recami, 2001; Steinberg, 2002)) and are confirmed by another approach also Perel'man (submitted).

Now the general approach to the von Neumann point of view is revised and generalized. With this purpose in the Section 2 the original idea of von Neumann is described and then in the Section 3 its strict mathematical expression is given, i.e. the adequate theory is constructed that allows the direct comparison of abstract properties of system with the admissible spectral representations. These representations have different forms including their conformity to equations of motion and Green functions. Some further possible generalizations are mentioned also.

In the Section 4 these constructions are applied to the simplest, one-dimensional cases, the Kramers–Kronig DRs and sampling theorems. On this base the special DRs for refractive indices are established. In the Section 5 corresponding many-point nonrelativistic DRs are considered.

The Section 6 is devoted to the 4-D relativistic DRs, their partial cases and so on. It will allow the continuing of analogue between the energy-momentum and time-space descriptions of scattering (Perel'man, submitted), they can be applied to generalizations of the Kramers–Kronig DRs by taking into account features of space dispersion also. In the Section 7 the partial integral representations of the 4-D covariant DRs are considered; their analyses allow to establish possibility and limits of nonlocal interactions, more precisely of the instantaneous transferring, close to limits of uncertainty principles, but slightly bigger them.

In the Section 8 the DR corresponding to the nonrelativistic Schrödinger equation is established on the base of exclusively kinematic considerations. This DR depends only on the single non-dimensional numerical parameter that is equally corresponding, *as can be assumed*, to all types of interactions. Therefore it seems interesting to compare this value with a hypothetical bare charge of SUSY, to which would approach the charges of all type interactions at such extremely high energies when all vacuum effects are vanishing, e.g. Weinberg (2000).

All considerations are briefly summed in the Conclusions.

2. THE STATEMENT OF VON NEUMANN

The formal theory of quantum properties can be constructed in such way.

Let us consider the complete Hilbert space of physical states $\mathbf{H}$. Each its closed subspace M specifies definite property of the considered system: the entire space is the sure property, and the orthogonal complement $M^{\perp}$ specifies the negation of the property. Two different properties define two subspaces M and N, therefore the four intersections $M \cap N$, $M^{\perp} \cap N$, $M \cap N^{\perp}$, and $M^{\perp} \cap N^{\perp}$ are each subspaces. If M and N are compatible, the disjunction of M and N is also defined and is the direct sum of the first three of the four subspaces; if M and N are noncompatible and their conjunction is impossible, then the properties are mutually exclusive.

Between subspaces of $\mathbf{H}$ and orthogonal projection operators (projectors, idempotent operators) there is a one-to-one correspondence: the subspace is the range of the projector P with $P^2 = P$. Therefore the projections I or 0 correspond to the sure property or to the impossible property, and the projection $P^{\perp} = I - P$ corresponds to the negation of the property specified by P. The properties associated with P and Q are compatible if P and Q commute and the projection $PQ = PQ$ represents the conjunction, when the projection $PQ + PQ^{\perp} + P^{\perp}Q = P + Q - PQ$ represents the disjunction. If $PQ = 0$, the properties are compatible, i.e. they are mutually exclusive, and the disjunction is represented by the sum $P + Q$.

Each property P provides a specification of a state probability $p(P)$. If there are projections $P_1, \ldots, P_n$ onto orthogonal subspaces with $P_1 + \ldots + P_n = I$, the additivity requirement is that $p(P_1) + \ldots + p(P_n) = 1$. Note that it is not necessarily assumed that every property can be measured or that all these probabilities are empirically meaningful: it is defined a mathematical specification of probabilities for all properties, whether they have physical meaning or not.

A pure state is determined by a unit vector of the Hilbert space ψ, the projection on the one-dimensional subspace spanned by ψ is $P = \psi \langle \psi, \ldots \rangle$. The probability of a property P when the system is in a pure state is given by the inner product $p(P) = \langle \psi, P\psi \rangle = \|P\psi\|^2 = \mathrm{tr}(PP)$, which is between 0 and 1. In general a state is defined as a pure state or as a randomized family of pure states.

All it shows the possibility of physical interpretation of properties by the geometry of corresponding projections.

3. MATHEMATICAL CONCRETIZATION OF THE STATEMENT

For better comprehension of suggested method we decompose it onto some subsections, in which will be given the general approach, its variants and some further perspectives will be mentioned.

3.1. Equations of Restrictions and their Integral Convolutions

If the conditions of system existence are determined by the precise ratio of parameters in the N-dimensional space of arguments, the corresponding projector δP, which can be considered as an atom of the Boolean algebra of projectors, must single out the hyperplane. If the admissible connection of arguments is extended onto the whole N-volume, such connection can be named non-holding and corresponding condition for the parameters would be named the equation of restriction (the notion of retaining liaisons in quantum theory was widely used by Dirac (1964)). Such subdivision is close to the more traditional in mechanics division of systems on holonomic and nonholonomic.

Let us consider more concretely a class of functions $\{f(x)\}, x \in R_n$ or $x \in E_n$, all of which can be attributed to two sublasses, $\{f_1(x)\}$ and $\{f_2(x)\}$, such that within the compact support S can be marked out subsupports characterized presence and absence of examined property:

$$\mathrm{supp} f_1(x) = S_1; \quad \mathrm{supp} f_2(x) = S_2. \tag{3.1}$$

These subsupports have, in general, common borders. Therefore they can be partially overlapped and their projectors can be non-orthogonal:

$$S = S_1 + S_2 - S_1 \cap S_2, \quad P_1 + P_2 - \delta P = 1, \tag{3.2}$$

where $\delta P = P_1 P_2$.

Orthogonal projectors can be introduced by exclusion a common part (common boundary) of supports:

$$\overline{P_1} = P_1 - \delta P; \quad \overline{P_2} = P_2 - \delta P, \quad \overline{P}_1 \overline{P}_2 = 0. \tag{3.3}$$

With such truncated projectors can be determined the relation:

$$\overline{P}_{1,2} f(x) = \overline{f}_{1,2}(x), \tag{3.4a}$$

the set $\{\overline{f}_{1,2}(x)\} \subseteq \{f_{1,2}(x)\}$, i.e. it corresponds to the properties described by $S_{1,2}$, but with possible absence of characteristics of boundaries.

By multiplication on projectors $\overline{P}_1$ or $\overline{P}_2$ the relation (3.4) can be rewritten as the equation of restrictions or as the condition of orthogonality:

$$\overline{f}_1(x) = \overline{P}_1 \ \overline{f}_1(x); \quad \overline{P}_2 \overline{f}_1(x) = 0. \tag{3.4b}$$

The functions with a support on the boundary,

$$\text{supp}\, \delta f(x) = S_1 \cap S_2; \quad \delta f(x) = \delta P \ f(x) \equiv P_1 P_2 f(x), \tag{3.5}$$

can be calculated or estimated separately and then added to both functions $\overline{f}_1(x)$ and $\overline{f}_2(x)$. Both conditions can be written together as the equations:

$$P_1(f - \delta f) = f_1 - \delta f \quad \text{or} \quad P_2(f_1 - \delta f) = 0, \tag{3.6}$$

where we return to initial projectors and which corresponds to subtraction proceduries.

The most required is the case when projectors can be expressed via the Heaviside unit functions, i.e. as $\theta(S_k(x))$. Along with them can be used the sign functions: $\text{sgn}(\xi) = \theta(\xi) - \theta(-\xi) = 2\theta(\xi) - 1$ and their generalization on the multidimension case in the form of the Riesz potentials: $\text{sgn}(\xi_\mu) \to \xi\mu / |\xi|$ (Stein, 1970).

Step functions are not determined on the boundary of supports, i.e. they correspond to the interior parts of domains. Hence a boundary can be often related to a space of lower dimension and therefore can be expressed through the delta-functions,

$$\delta f(x) = \delta(S_1 \cap S_2) \ f(x), \tag{3.7}$$

and their derivatives.

If it can be suggested that the considered functions allows the Fourier transformations (FT, the transformations of other types can be also used) over all variables, the FT of projectors $\widehat{\mathbf{F}}[P_i]$ must be calculated, usually in the sense of generalized functions, then the corresponding equations of restriction will be represented as the integral convolutions:

$$\overline{f}_1(k) = \widehat{\mathbf{F}}_k[\overline{P}_1] \bigotimes \overline{f}_1(k); \quad \widehat{\mathbf{F}}_k[\overline{P}_2] \bigotimes \overline{f}_1(k) = 0; \tag{3.8}$$

$$\delta f(k) = \widehat{\mathbf{F}}_k[P_1 P_2] \bigotimes f(k). \tag{3.9}$$

(the FT of functions are denoted by the same symbols with different arguments: $x \to k$, etc.). The determination of projectors shows that

$$\widehat{\mathbf{F}}_k[\overline{P}_{1,2}] = \frac{1}{2}\delta(\dots) \pm Q(k), \tag{3.10}$$

and, correspondingly,

$$\overline{f}_{1,2}(k) = \pm 2 \int dq\, Q(k-q)\, \overline{f}_{1,2}(q) \tag{3.11a}$$

or

$$\overline{f}_1(k) \pm \overline{f}_2(k) = 2 \int dq\, Q(k-q)\{\overline{f}_1(q) \mp \overline{f}_2(q)\}. \tag{3.11b}$$

As $\overline{f}_1 - \overline{f}_2 = f_1 - f_2$ these equations can be expressed in another forms also.

Note that the duality of (3.11) demonstrates, in general form, the principle of J. Babinet: the main and additional picture of diffraction and so on are equivalent.

These results can be formulated for functions of Euclidian (R_n) or pseudo-Euclidian (E_n) set of variables and of definite classes of integration L_p, provided existence of integral transformations, as

Theorem 1. *A class of functions $f(x) \in L_p$ (R_n or E_n), $0 < p < \infty$, characterized by the compact support S_k, is described, with taking into account necessary regularizations, by equations of restrictions and/or by the equivalent to them identities of orthogonality (3.6) and corresponding equations in integral convolutions of the types (3.8), (3.11).*

If the kernel $Q(k)$ is a real function and $\overline{f}(k)$ is a complex one, the relations (3.11) allows to express real part of $\overline{f}(k)$ through the imaginary part or vice versa. Such relations are named dispersion relations, DRs. If $Q(k)$ is a complex function, it leads to integral equations connecting both parts of $\overline{f}(k)$ or to spectral representations, but we will use in all cases the name "DR" for brevity.

The integral representation (3.9) can be refined if the support δS of examining class of functions is reduced to hypersurface of lower dimension. For such cases follows

Theorem 2. *The class of functions, which are equal zero outside the hypersurface $\delta S(x)$, can be represented after needed regularizations by their FT as*

$$f_{\delta S}(k) = \int dx \phi(x)\, \delta[P_{\delta S}(x)]\, e^{i(k,x)}, \tag{3.12}$$

where $\phi(x)$ is an arbitrary function non–singular on the borders of support. Functions (3.12) satisfy the homogenious pseudodifferential equation

$$P_{\delta S}(-i\partial_k)\ f_{\delta S}(k) = 0. \tag{3.13}$$

Function $\phi(x)$ and limits of integration can be determined by initial and borderline conditions.

As a natural generalization of this representation the Green function of non-homogenious analog of (3.13) can be determined:

$$G^{(\pm)}_{\delta S}(x)\ \delta[P_{\delta S}(x)] = 1. \tag{3.14}$$

The FT of this function

$$G^{(\pm)}_{\delta S}(k) = \int dx\ \gamma(x)\ \delta_{\pm}[P_{\delta S}(x)]\ e^{i(k,x)}, \tag{3.15}$$

and if x_n are roots of the equation $P_{\delta S}(x) = 0$, the norms of all $\gamma(x_n)$ are equal 1 or 0, i.e. all $\gamma(x_n)$ are expressed via θ-functions.

3.2. Representations via Green Functions

The different approach for establishment of DRs is based on the general possibility of decomposition of every function over the set of self-functions of the problem. Such approach will reveal the principal accordance of DRs and equations of motion.

Let some function $\Phi_S(x) = \xi$ completely covers the supp$\{f(x)\} = P_S$, when ξ run along $[0, \infty)$,

$$\int_0^{\infty} \delta(\Phi_S(x) - \xi)\, d\xi = \theta(\Phi_S(x)) \equiv P_S(x), \tag{3.16}$$

and can be considered as the symbol of pseudodifferential operator:

$$\Phi_S(-i\partial_p)\ f(p) = \int dx e^{i(p,x)} \Phi_S(x)\ f(x). \tag{3.17}$$

The FT of this projector can be represented as

$$P_S(p) = \int dx e^{i(p,x)} P_S(x) \to \int_0^{\infty} d\xi\ G^{(0)}_{\xi}(p), \tag{3.18}$$

where

$$G^{(0)}_{\xi}(p) = \int dx e^{i(p,x)} g(x)\ \delta(\Phi_S(x) - \xi) \tag{3.19}$$

is the Green function of a homogenious equation,

$$[\Phi_S(-i\partial_p) - \xi] G^{(0)}_{\xi}(p) = 0, \tag{3.20}$$

represented in the form of Radon transform; $g(x)$ is an arbitrary function nonsingular at borderlines.

Hence the FT of equation of restriction, $f(x) = P_S(x)\, f(x)$, can be presented in the convolution form:

$$f(p) = (2\pi)^{-n} \int_0^{\infty} d\xi\, G_{\xi}^{(0)}(p) \bigotimes_p f(p). \tag{3.21}$$

It shows, in particular, that

$$\Phi_S(-i\partial_p) f(p) = (2\pi)^{-n} \int_0^{\infty} d\xi\ \xi G_{\xi}^{(0)}(p) \bigotimes_p f(p). \tag{3.22}$$

The possible decomposition

$$f(p) = \int_0^{\infty} d\xi\ f(p, \xi), \quad [\Phi_S(-i\partial_p) - \xi]\, f(p, \xi) = 0 \tag{3.23}$$

shows that functions representing the property S can be considered as the direct sum of solutions of specific differential equations.

It can be formulated as

Theorem 3. *The class of functions with property S can be represented by their total sum with single Green functions of equation describing this property at partial choice of parameters, which represent the expansion of arbitrary function over self-functions of problem. If this equation represents the equation of motion, such integral convolution allows a specialization of DRs by taking into account the properties of propagators.*

Notice that instead an introduction of additional parameter ξ such coefficient in $\Phi_S(x)$, at changing of which the considered hyperplane run through the whole volume of needed support, can be used. It reveals the conformity of (3.18) with the Radon transformation, in which the decomposition goes over natural for the considered problem hypersurfaces of equations of motion (cf. Gel'fand *et al.* (1962) where are introduced artificial orispheres or orbits).

3.3. Functions of Many Variables and their Incomplete Transformation

For functions of many variables the suggested method allows additionally such interesting possibilities.

Let's consider the partial FT of (3.6), for example, relative the first $(n-1)$ variables of $f(x_1, \ldots, x_n)$. Thus identically

$$\int dq_1 \ldots dq_{n-1} \overline{P}_2(q_1, \ldots, q_{n-1}|x_n)\, \overline{f}_1((k-q)_1, \ldots, (k-q)_{n-1}|x_n) = 0, \tag{3.24}$$

where the non-transformed variable is separated by vertical line. The equality (3.24) would hold at arbitrary magnitudes of continuous variables k_i and x_n. Hence, its integrand should be identically equal zero:

$$\overline{P}_2(q_1, \dots, q_{n-1}|x_n)\ \overline{f}_1((k-q)_1, \dots, (k-q)_{n-1}|x_n) = 0, \tag{3.25}$$

Here can be underlined an analogue of this transition with a variation of function (integral) of action for continuous media and transition to the basic canonical equations with the Lagrangian density.

In view of the fundamental definition $\xi \times \delta(\xi) = 0$ it is evident that if a function $f(\xi)$ is non-singular, and its (isolated) zeros in points ξ_m are not higher n-th order, the general solution of an algebraic equation $P(\xi)\ f(\xi) = 0$ with the known function $P(\xi)$ has a kind:

$$f(\xi) \sim \delta(P(\xi)). \tag{3.26}$$

Therefore the formal solution of (3.25) can be written down as

$$\overline{f}_1(k_1, \dots, k_{n-1}|x_n)) = \phi(k_1, \dots, k_{n-1}|x_n)\ \delta(\overline{P}_2(q_1, \dots, q_{n-1}|x_n)), \tag{3.27}$$

with arbitrary function ϕ such that its zeros and poles do not coincide with zeros of $\overline{P}_2$. Thus the δ-function must be decomposed as the function of magnitudes x_n only and if $\overline{P}_2^{(n-1)}(\xi) = 0$, $\overline{P}_2^{(n)}(\xi) \neq 0$, then

$$\delta[\overline{P}_2(\xi_s)] = \sum_s \left|\overline{P}_2^{(n)}(\xi_s)\right|^{-1} \delta^{(n-1)}(\xi - \xi_s) \to \sum_0^n \sum_s a_{n,s} \delta^{(n)}(\xi - \xi_s), \tag{3.28}$$

with arbitrary coefficients $a_{n,s}$ that must be determined by other reasons or models. (Note that (3.27) can be considered as the pointed equation of restriction. The correspondence between such approach and the theory of residues seems obvious and enables the substantiation of suggested approach by the more traditional methods.)

This construction can be formulated as

Theorem 4. *If the FT of projector over some variables allows decomposition on simple multipliers over non-transformed variables, thus transformed functions have the form (3.27–28) with arbitrary coefficients.*

Let's note that the substitution $P'(x) = 1 - P(x)$ in the (3.24) leads to the DRs with participation of partially Fourier transformed functions, which, possibly, does not have so simple formal solution, but reveals, basically, some correspondence between allowable changes of variables. Thus, the suggested method results in determination or, at least, in specification of dependence of functions of a researched class on a part of arguments.

3.4. Additional Possibilities

There are certain possibilities of suggested method which we did not use and which would be briefly mentioned.

1. Till now we had not used the main property of projectors: $P^2(x) = P(x)$. This property allows rewriting, for example, the single equation of restriction as

$$f_S(p) = \int dq \int dq_1 P_S(p - q_1) P_S(q_1 - q)\, f_S(q)$$
$$= \bigotimes_1^2 P_S(p) \bigotimes f_S(p). \tag{3.29}$$

In such form it is a simple identity, but if in the condition of the theorem 3 into FT of different projectors may be inserted different Green functions or can be used different limits of integration, then these representations can be usable for description of complicating processes with scattering on several centres and so on.

Note that such iteration can be continuing infinitely:

$$f_S(x) = \lim_{N\to\infty} P^N(\varphi(x))\, f_S(x), \tag{3.30a}$$

which after transition to the Fourier convolution leads to a formal analog of the continual integral:

$$f_S(p) = \lim_{N\to\infty} \bigotimes_0^N \int dq_n \widehat{\mathbf{F}}[P(\varphi(q_n - q_{n-1}))]\, f_S(q_1), \quad q_N \to p. \tag{3.30b}$$

2. We still did not consider such properties of physical systems as their symmetries. The symmetries introduce clarity and sometimes simplifications into considered DRs, but there is also another and more general possibility. Let's show these possibilities.

An operation of symmetry in the quantum theory means the transformation of vectors in the Hilbert space: $\Phi, \Psi \in \mathbf{H}$ are transformed into $\Phi', \Psi' \in \mathbf{H}$ or $\mathbf{H}'$ with preservation of probabilities of all processes:

$$|(\Phi', \Psi')|^2 = (S\Phi, S\Psi)|^2 = |(\Phi, \Psi)|^2. \tag{3.31}$$

Superselection rules single out in $\mathbf{H}$ coherent subspaces. If an operator S reflects one subspace on another or on the itself, then in accordance with the Wigner theorem (e.g. Streater and Wightman, 1964) it can be only unitarily or antiunitarily, at that it is an operator of complete or partial isometry.

Let's consider as an example the Lorentz transformations. Their covariance in the **H** means that

$$(\Phi, \Psi)_{\mathbf{H}} = (\Lambda\Phi, \Lambda\Psi)_{\Lambda\mathbf{H}}\,. \tag{3.32}$$

As Λ is the operator of partial isometry, $\Lambda^{+}\Lambda = P_{\Lambda}$ is the projector and (3.32) can be rewritten as

$$||P_{\Lambda}\Psi|| = ||\Psi||\,, \tag{3.33}$$

i.e. as the equation of restrictions in the weak topology. The transition to FT leads, naturally, to DRs in the weak topology, which we do not consider here.

Underline that such DRs can relate to a set of objects as the operator $\mathbf{P_F} = \mathbf{F}^{+}\mathbf{F}$ of FT can also be considered as the operator of symmetry. The operator of conjugation can be presented as $\mathbf{I} = \mathbf{UC}$, where **U** is an operator of involution and **C** is an operator of the Hermite conjugation. Such generalization allows a consideration of such complicated combinations as the **CPT** symmetry. Moreover, on this base can be considered, generally speaking, the gauge transformations also.

4. ONE-DIMENSIONAL RELATIONS

There are two types of relations of considered type: the restrictions on half-axis and restrictions on finite interval. The first type leads to the Kramers–Kronig DRs (possibilities of their generalization on the space dispersion phenomena is mentioned in the Section 6) and the second type leads to the sampling theorems and their generalizations. (Sampling theorems have a big and very tangled history (Jerri, 1977; Läke, 1999), in these articles are cited also their numerous applications in physics, etc.).

4.1. Kramers–Kronig Dispersion Relations

Usual deduction of DRs for transient (response) functions is executing by such scheme. If the incoming and outlet signals or components of fields strengths (possible tensor indices are omitted) are connected by the linear relation:

$$O(t, \mathbf{r}) = \int dt' d\mathbf{r}'\; R(t - t'; \mathbf{r} - \mathbf{r}')\; I(t', \mathbf{r}') \equiv R(t; \mathbf{r}) \bigotimes_{t,\mathbf{r}} I(t, \mathbf{r}), \tag{4.1}$$

the principle of (primitive) causality requires the strict execution of the condition: $R(t; \mathbf{r}) = 0$ at $t < 0$ (this condition does not touch its magnitude at $t = 0$). If the existence of partial FT of $R(t; \mathbf{r})$ can be assumed, then $R(\omega, \mathbf{r})$ is an analytic function in the upper half-plane and, in accord with the famous Titchmarsh theorem 95 (Titchmarsh, 1948), satisfies the Hilbert relations.

But the equation (3.8) with $\overline{P}_{1,2} = \theta(\pm t)$ and $f(k) \to R(\omega, \mathbf{r})$ we can write without examination of analitical features:

$$\theta(-t)R(t; \mathbf{r}) = 0. \tag{4.2}$$

Hence its FT over t gives

$$R(\omega, \mathbf{r}) = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{d\eta}{\omega - \eta} R(\eta, \mathbf{r}), \tag{4.3}$$

where P is the symbol of the Cauchy principal value of integral (below it will be omitted). For transition to the Kramers–Kronig DRs the symmetrization of functions must be taken into account.

The formal differentiation of (4.2), i.e. the assumption about the existence and physical sense of derivatives of response functions, leads to the relations:

$$\theta(-t)R^{(n)}(t; \mathbf{r}) = -\sum_{k=0}^{n} \delta^{(k)}(t) R^{(n-k)}(t = 0; \mathbf{r}), \tag{4.4}$$

where, for example, at transient processes in electrical circuits $n = 2$, $R'(t = 0)$ and $R''(t = 0)$ characterize variations of voltage on capacitors and of current in inductors, correspondingly, i.e. they describe the inertial features of circuits.

The FT of (4.4) means executing of such DR for arbitrary $n \geq 0$:

$$\omega^n R(\omega) = \frac{1}{\pi i} \int_{-\infty}^{\infty} d\eta \frac{1}{\omega - \eta} \eta^n R(\eta, \mathbf{r}) - 2\, i^{-n} \sum_{k=0}^{n} (i\omega)^k R^{(n-k)}(t = 0), \tag{4.5}$$

additional terms of which can describe several inertial features of media.

Note that the condition (4.2) allows the substitution: $R(\omega) \to t^M R(t)$ with arbitrary non-negative M, even non-integer. Such substitution leads to the DRs (4.3) for derivatives $R^{(M)}(\omega)$, but without additional substraction terms which can be reestablished, if needed, by subsequent integration. Such DRs represent the smoothed parameters of considered processes. Along with such substitution some others are also possible, so, for example, into these relations can be inserted FT of $\ln[1 + R(t)]$.

However the basic relation (4.1) allows the analyses of transient characteristics, e.g. the components of dielectric susceptibility $\varepsilon_{ij}(t, \mathbf{r})$, but as the index of refraction $N(\omega) = \sqrt{\varepsilon(\omega)}$, the direct use of considered procedure is, in principle, impossible, although it is often suggested and used.

For these reasons we consider more general approach to such problems. So, if the distance between the incoming and outlet points of signal is fixed,

$$O(t, x + L) = \int dt' R(t - t'; L) I(t', x), \tag{4.6}$$

the condition of general causality (4.2) can be rewritten as

$$\theta(-t)R(t + L/c; L) = 0. \tag{4.7}$$

Let's, for simplicity, assume that the divergence of transferred signal is neglected. If an intermediate medium is uniform and continuous, it can be considered as systems of layers with neglecting reflections on their borders, and therefore it can be proposed that the FT of response functions exponentially depends on distances:

$$R(\omega; L) = \int dte^{i\omega t} R(t; L) = \exp(-\varphi(\omega)L) \tag{4.8}$$

and $R(\omega; L = 0) = 1$; complex functions $\varphi(\omega) = \varphi_1 + i\varphi_2$ can depend on wave numbers, structure of medium, intensity of light, external fields, but all these dependences can be here omitted. In line with the common physical reasonings φ_1 must be identified with the coefficient of absorption and just therefore φ_2, correspondingly, must be proportional to the index of refraction: $\varphi(\omega) \rightarrow \kappa(\omega) + in(\omega)$.

The substitution of (4.8) into the DR (4.3), decomposition of exponents in both sides over L and equating of terms with equal degrees of L lead, in particular, to the Kramers–Kronig relations for the index of refraction:

$$n(\omega) - 1 = \frac{2}{\pi} \int_0^\infty \frac{\eta d\eta}{\omega^2 - \eta^2} \kappa(\eta). \tag{4.9}$$

The integration of (4.9) in the infinite limits leads to the sum rules for optical parameters of scatterers:

$$\int_{-\infty}^{\infty} d\omega(n(\omega) - 1) = 0, \tag{4.10a}$$

established in Saslow (1970); Altarelli *et al.* (1972).

Terms of exponent expansion at L^2 lead to the DR for $\varepsilon(\omega) = \varphi(\omega)^2$, other terms of decomposition give more complicate relations. As the generalization of (4.10) they lead to the sum rules:

$$\int_{-\infty}^{\infty} d\omega\, \omega^m (n(\omega) - 1)^m = 0, \tag{4.10b}$$

i.e. to so named superconvergent rules for indices of refraction, which were deduced in Altarelli and Smith (1974). Our deduction evidently demonstrates an approximate character of these relations and gives possibilities of their refinements.

As our deduction is not connected with the Cauchy theorem, i.e. with the closing of integration contour, it can be applied, in principle, equally to passive and active systems. It means that the Kramers–Kronig relations do not depend themselves from the condition of passivity (cf. Wang, 2002).

Notice that unlike the usual deduction of DRs the proposed method allows the determination of DRs for non-uniform media with variable index of refraction $n(x)$. For this aims by the substitution

$$\varphi(\omega)L \to \int_0^L \frac{\partial}{\partial x}\varphi(\omega, x)\,dx \tag{4.11}$$

in the response function we shall come to DRs for corresponding integrals instead index of refraction.

4.2. Sampling Theorems

If the signal duration is strictly limited by the interval $(-T, T)$, i.e. the response function $f(t)$ is equal zero at $|t| > T$, this condition can be recorded as the equation of restrictions or of orthogonality:

$$f(t) = \theta(T^2 - t^2)\, f(t) \quad \text{or} \quad \theta(t^2 - T^2)\, f(t) = 0. \tag{4.12}$$

Their FT leads to the Duhamel integral:

$$f(\omega) = \frac{1}{\pi}\int \frac{d\eta}{\eta}\, f(\omega - \eta)\sin T\eta, \tag{4.13}$$

which expresses the value of real function in one point via its values in whole line. Usually there are used the notation $\sin\xi/\xi = \text{sinc}\xi$.

By transformation with the Fourier series instead the integral, possible in view of increasing convergences, the series becomes

$$f(\omega) = \frac{1}{\pi T}\sum f(\eta_k)\sin T(\eta_k - \omega)/(\eta_k - \omega). \tag{4.14}$$

This expression, named usually the Shannon theorem, allows signals reconstruction by a little number of counts and is widely used in the information theory, radioscience, etc. It can be considered as the finite interval DR.

With the substitution $(t, \omega) \to (x, k)$ such integrals or sums naturally describe wave transferring through a slit, by introduction of the sum of projectors with different space location they can be adopted to consideration of interference pictures, etc. The generalization of this method on a multidimensional cases seems evident. Note that at definite symmetry of problem the more suitable transformation can be used. So, at the wave transfer through circular hole the Bessel transformation will the most adequate, in the 3-D case the Gabor series are useful.

The differentiation of (4.12) leads, as in the case of (4.5), to the taking into account the boundary conditions and their features.

If the transparence of sample is different on various frequencies of the used band, then such equation can be written:

$$O(\omega) = \theta(\omega_0 - |\omega|)Q(\omega)I(\omega), \tag{4.15}$$

where $Q(\omega)$ is the function of transparency.

There can be considered many types of various devices. So if $Q(\omega) = \omega_0 - \omega$ and for the sake of simplicity $\omega_0 = 1$, the transformation of (4.15) leads to the so named Watson transformation:

$$f(t) = (1/\pi^2) \int dt' \; F(t' - t)\, t'^{-2} \sin^2(t'/2) \tag{4.16}$$

that converges rapidly (4.13) and requires lesser number of samplings. The best result becomes for diffraction on a hole with $Q(\mathbf{r})$ in the form of the Gauss (normal) distribution: the diffracted rays would be absent at all, such method is named the apodisation in optics.

5. MULTICHANNEL DEVICES AND MULTIPHOTON PROCESSES

There are many attempts of DRs generalization on the higher order response functions including multiphoton processes (e.g. Peiponen *et al.* 2004) and references therein). But they did not lead to practically usable results.

Let's consider ab initio possibilities of their deduction for a nonlinear system:

$$O(t) = \int dt_1 \cdot \ldots \cdot dt_N \; f(t - t_1, \ldots, t - t_N)\, I_1(t_1) \cdot \ldots \cdot I_2(t_N). \tag{5.1}$$

Necessary and sufficient conditions of general causality of the system are expressed by the system of N equations of restrictions:

$$f(t_1, \ldots, t_N) = \theta(t_k)\, f(t_1, \ldots, t_N) \qquad \{k = 1, \ldots, N\}. \tag{5.2}$$

Possible compositions of these equations lead to representations:

$$f(t_1, \ldots, t_N) = \frac{1}{N} \sum_1^N \theta(t_k)\, f(t_1, \ldots, t_N); \tag{5.3}$$

$$f(t_1, \ldots, t_N) = \frac{1}{N(N-1)} \sum_{i \neq k}^N \theta(t_i)\theta(t_k)\, f(t_1, \ldots, t_N); \ldots$$

$$f(t_1, \ldots, t_N) = \prod_1^N \theta(t_k)\, f(t_1, \ldots, t_N). \tag{5.4}$$

The substitutions $\theta(t_k) = 1 - \theta(-t_k)$ show that the last relation represents the sum of all preceding ones. Each of them, except the last one, is necessary, but can be non-sufficient for complete fulfilment of the causality requirements.

These representations are very complicated, but their analyses can begin with the simplest one (5.3) with further checking by other relations.

The most complete and the most complicate representation (5.4) can be transformed in a more compact form (Perel'man, 1971). Let's consider for this purpose its FT:

$$f(\omega_1, \dots, \omega_N) = \int \prod_1^N d\eta_k \; \delta_+(\omega_k - \eta_k) \; f(\eta_1, \dots, \eta_N). \tag{5.5}$$

For processes in the pure monochromatical field the substitution of type

$$f(\eta_1, \dots, \eta_N) \to f(\overline{\eta}, J) \prod \delta(\overline{\eta} - \eta_k) \tag{5.6}$$

or with more real frequencies distributions are possible. It will leads to definite series which can be, in principle, summed.

If $f(t_1, \dots, t_N)$ is real, such representation for its FT, by taken into account the conditions of symmetry, can be written:

$$Re\ f(\omega_1, \dots, \omega_N) = (2\pi i)^n \rho_1\big(\omega_1^2, \dots, \omega_N^2\big), \tag{5.7a}$$

$$Im\ f(\omega_1, \dots, \omega_N) = (-2\pi i)^n \prod_k sgn(\omega_k) \rho_2\big(\omega_1^2, \dots, \omega_N^2\big). \tag{5.7b}$$

With the Green function of oscillator, $G_0^{(\pm)}(\omega, \mathbf{q}^2) = (\omega^2 - q^2 \pm i\omega_{\to 0})$, the multi-frequencies DRs can be represented as a multidimensional generalization of the known Lehmann spectral representation for fermions (e.g. Bogoliubov and Shirkov, 1982):

$$f(\omega_1, \dots, \omega_N) = \prod_1^N \int_0^\infty dq_k^2 G_0^{(-)}\big(\omega_k, \mathbf{q}_k^2\big) \times \left\{ \rho_1\big(q_1^2, \dots, q_N^2\big) + i \prod_k \mathrm{sgn}(\omega_k) \rho_2\big(q_1^2, \dots, q_N^2\big) \right\}. \tag{5.8}$$

This representation allows to construct and check different models.

More simple and practical seems such approximations: the transition to new N independent variables in (5.4): $T = (t_1 + \cdots t_N)/N$ and $\{\tau_{i,k} = t_i - t_k\}$ with decomposion of DR over $\tau_{i,k}$ as functions of J and ω. More generally to these DRs can be applied the condition or property of temporal compatibility of signals or photons. Such property can be expressed by the factor $g(\tau_{i,k}|\tau_0)$, where τ_0 is the waiting time of device or the duration of keeping of energy of first photons, virtually absorbed by scatterer, till absorption of subsequent ones. (Notice that this function represents in some sense an analog of the Bogoliubov function of switching on interaction (Bogoliubov and Shirkov, 1982; Perel'man, submitted).)

It leads to the substitution:

$$f(t_1, t_2) \to f_\tau(t_1, t_2) = g(\tau_{1,2}|\tau_0)\, f(t_1, t_2) \tag{5.9a}$$

with the FT

$$f(\omega, \eta) \to f_\tau(\omega, \eta) = \int d\xi\ g(\xi|\tau_0)\, f(\omega - \xi, \eta + \xi) \tag{5.9b}$$

in the DR (5.3) and others.

For the switching function several forms can be suggested, the simplest among which are

$$g_1(t|\tau) = \theta(\tau^2 - t^2), \quad g_2(t|\tau) = \exp(-|t|/\tau), \quad g_3(t|\tau) = \exp(-t^2/\tau^2). \tag{5.10}$$

If for technical devices τ can be a constant magnitude, in multiphoton processes $\tau = \tau(\omega, J)$: it depends on the field power J and till definite threshold value $\tau \sim 1/J$, at bigger J it can be assumed that $\tau \sim 1/\omega$.

Note that the usage of g_1 leads to a peculiar symbiosis of Kramers–Kronig DRs and sampling theorem. But they all do not allow, in general case, such division of relations where the right side depends on real or imaginary parts only. Therefore these relations can have lesser applications than one-particle DRs or sampling theorems.

The most interesting feature of corresponding DRs consists in its dependence on τ: it can give possibility of examining models or calculations of $\tau(\omega, J)$, i.e. the duration of interaction.

6. EQUATIONS OF RESTRICTIONS AND SPECTRAL REPRESENTATIONS IN THE E_4-SPACE

We consider 4-hyperboloid in the pseudo-Euclidian space E_4 with projector of inner domain $P = \theta(t^2 - r^2 - s^2)$, $t \in R_1$, $r \in R_3$, $s^2 =$ const. This projector and corresponding equations can be examined in the total FT $(\omega; \mathbf{k})$ and in the partial FTs $(t|\mathbf{k})$, $(\omega|\mathbf{r})$, $(t; x, y|k_z)$. These different representations reveal various physical features and therefore the partial FT will be considered in the next Section.

The projector of relativistic (or Einsteinian) causality leads to the equation of restrictions:

$$f(t, \mathbf{r}) = P_{RC}^{(\mathrm{adv})} f(t, \mathbf{r}) = \theta(t)\theta(t^2 - r^2)\, f(t, \mathbf{r}). \tag{6.1}$$

This projector is simplified in the 2-D case: $P_{RC}^{(\mathrm{adv})}(t, x) = \theta(t - x)\theta(t + x)$, and the condition of causality can be rewritten for reduced space by the system of equations:

$$f(t,\mathbf{r}) = \theta(t-x)\, f(t,\mathbf{r}); \tag{6.2a}$$

$$f(t,\mathbf{r}) = \theta(t+x)\, f(t,\mathbf{r}), \tag{6.2b}$$

each of which is necessary, but unsufficient for satisfiability of causality condition requirements. In the 4-D case two equations for $f(t,\mathbf{r})$ with projectors $\theta(t)$ and $\theta(t^2 - r^2)$ can be separately written, i.e. as the system of Kramers-Kronig DRs and 3-D sampling theorem, but it does not simplify their consideration.

Complete FT of hyperboloid projector was in detail studied in Perel'man (1966) and on some another base will be written below. It must be only noted that the FT of complicate projectors required definite cautions as the operators of transformation over different variables do not commute in general. So the FT of projector over t, then over space variables and after it in the inverse order lead to different results ($s^2 = 0$ for simplification):

$$\widehat{\mathbf{F}}_{\mathbf{k}}\widehat{\mathbf{F}}_{\omega}[\theta(t)\theta(t^2-r^2)] = \frac{1}{2}\delta(\omega)\delta(\mathbf{k}) - (1/8\pi^2 i\omega k)\,\partial_{\mathbf{k}}\,\{\delta_+(\omega-k)+\delta_-(\omega-k)\}\,; \tag{6.3a}$$

$$\widehat{\mathbf{F}}_{\omega}\widehat{\mathbf{F}}_{\mathbf{k}}[\theta(t)\theta(t^2-r^2)] = -(1/4\pi^2 i\omega k)\,\partial_{\mathbf{k}}\,\{\delta_+(\omega-k)+\delta_-(\omega-k)\}\,. \tag{6.3b}$$

These difference still corresponds to the Poincare-Bertrand formulae for permutation of integration order in singular integrals (e.g. Stein, 1970), therefore the equations of restriction are in both cases identical.

The projectors of E-$\mathbf{p}$-hyperboloid can be constructed analogically. So, for the states with positive energy only

$$f(E,\mathbf{p}) = P_E^{(+)} f(E,\mathbf{p}) \equiv \theta(E)\,\theta(E^2-p^2-m^2)\, f(E,\mathbf{p}). \tag{6.4}$$

More obvious seems the representation

$$P_E^{(+)}(E,\mathbf{p}) = \theta(E)\int_{m^2}^{\infty} d\mu^2 \delta(E^2-p^2-\mu^2) \equiv \frac{1}{(2\pi)^3 i}\int_{m^2}^{\infty} d\mu^2 \Delta^{(+)}(E,p,\mu), \tag{6.5a}$$

where $\Delta^{(+)}(E,p,\mu)$ is the positive frequence Green function of homogenious Klein-Gordon equation. It means that DRs based on the axioms of mass-spectrality and positivity of energy,

$$f(E,\mathbf{p}) = \frac{1}{(2\pi)^3 i}\int_{m^2}^{\infty} d\mu^2 \Delta^{(+)}(E,p,\mu) \bigotimes\nolimits_{E,\mathbf{p}} f(E,\mathbf{p}), \tag{6.5b}$$

represent the set of all "masses" (virtual energies states) admissible by these axioms with equal probabilities, possible restrictions of its spectrum, the account of resonances and so on can refine these relations.

The projector of relativistic causality can be considered via analogical construction:

$$P_{RC}(t, \mathbf{r}) = \theta(-t) \int_0^\infty ds^2 \delta(t^2 - r^2 - s^2) \equiv \frac{1}{(2\pi)^3 i} \int_0^\infty ds^2\, \nabla^{(\mathrm{ret})}(t, \mathbf{r}, s), \tag{6.6a}$$

where $\nabla^{(\mathrm{ret})}(t, r, s)$ is the retarded Green function of the reciprocal Klein–Gordon equation,

$$\{\partial^2/\partial E^2 - \partial^2/\partial \mathbf{p}^2 - s^2)\nabla^{(\cdot)}(E, \mathbf{p}; s) = 0, \tag{6.6b}$$

determined by the usual Green functions with substitutions $(t, \mathbf{r}, s) \leftrightarrows (E, \mathbf{p}, m)$ (Perel'man, 1966).

The corresponding DRs would be analogical to (6.5b):

$$f(t, \mathbf{r}) = \frac{1}{(2\pi)^3 i} \int_0^\infty ds^2 \nabla^{(\mathrm{ret})}(t, \mathbf{r}, s) \bigotimes_{t,\mathbf{r}} f(t, \mathbf{r}). \tag{6.7}$$

The operator $\tau = \partial / i \partial E$ and correspondingly the operator $\vec{\rho} = i\partial/\partial \mathbf{p}$ describe the duration and the space extent of interactions (Perel'man, submitted). Therefore the physical base of (6.7) can be symbolically represented as

$$\tau^2 - \vec{\rho}^{\,2} = s^2, \tag{6.8}$$

i.e. as the reciprocal one to $E^2 - p^2 = m^2$, this relation does not determine the sign of τ. Hence the relativistic DRs can be considered as a set of self-functions of (6.8) with nonrestricted "4-intervals of interaction" s. Note that operators of (6.8) types are used at construction of a general kinetic equation but without suggested interpretation.

It must be here underlined that the relativistic DRs in their retarded form do not determine the interactions completely. For this purpose are needed also the relations based on the spectrality of (6.5b) type. Impossibility of theory construction on the base of only retarded interaction was marked for the first time, as far as I know, by Einstein in the discussions with Ritz (Ritz and Einstein, 1909).

Note here that the general forms of DRs do not allow complete separation of real and imaginary parts of response functions and represent the system of integral equations for them. The single linear relations of the (6.2) type, taken one by one, can lead to some results, which after must be checked by more complete relations (Perel'man and Englman, 2000). (Notice that in the article (Perel'man, 1966) we marked this unsufficientness of the theory suggested in Leontovich (1961) on the base of only one equation (6.2), but this suggestion is from time to time repeated, e.g. Thoma, 2000).

Nevertheless in optical applications some simple approximations to (6.5a) can be considered. So these DRs are satisfied by the substitution of dielectric susceptibility: $f(\omega,\mathbf{k}) \to \varepsilon(\omega,\mathbf{k}) = \varepsilon(\omega) + \sum a_n(\omega)k^n$; for media with natural optical activity can be taken that $\varepsilon(\omega,\mathbf{k}) \approx \varepsilon(\omega) + k\alpha(\omega)$, for non gyrotropic media $\varepsilon(\omega,\mathbf{k}) \approx \varepsilon(\omega) + k^2\beta(\omega)$ for which the refined Kramers-Kronig DRs can be written. It can be taken into account that space dispersion has place in definite intervals of frequencies only, the form of poles to transparent media can be suggested and so on. But their concrete consideration is sufficiently far from our aims in this paper.

7. LOCAL AND NONLOCAL INTERACTIONS, TUNNELING

The representation $(t|\mathbf{k})$, i.e. the FT of light cone projector over $\mathbf{r}$, is of form:

$$P(t|\mathbf{k}) \equiv \widehat{\mathbf{F}}_{\mathbf{k}}[P(t,\mathbf{r})] = (1/2\pi)^3 \int d\mathbf{r} e^{i\mathbf{kr}}\theta(t^2 - r^2)$$
$$= (1/2\pi^2k^3)\{\sin(kt) - kt\cos(kt)\}. \tag{7.1}$$

At $t \to \infty$ the projector $P(t,\mathbf{k}) \to \delta(\mathbf{k})$, at $kt \to 0$ it is represented by series:

$$P(t|\mathbf{k}) \to (1/\pi^2k^3)\sum_1^\infty (-1)^{n+1}(kt)^{2n+1}n/(2n+1)!. \tag{7.2}$$

The substitution $\theta(t^2 - r^2) \to \theta(t^2 - r^2 - s^2)$ leads to the same expressions with $t \to (t^2 - s^2)^{1/2}$.

At consideration of δ-function of (7.1) all irrelevant terms must be factoring, hence it will be transformed to $\delta(\tan(kt) - kt)$. First numerical solutions of the transcendental equation $\tan\xi = \xi$ are equal 0; 4.493; 7.725; ... (approximately), in which connection the first zero at $t = 0$ is of the third order.

As $\xi^m\delta^{(n)}(\xi) = 0$ at $n \le (m-1)$, the general solution of the equation of (3.21) type, i.e. all function converting into zero inside hyperboloids (non-local functions) are of the kind:

$$R_{NL}(t|\mathbf{k}) = A\delta(P(t|\mathbf{k})) = \varphi_0\delta(t) + \varphi_1\delta'(t) + \varphi_2\delta''(t) + \sum_n \varphi_n\delta(t - t_n) + \cdots \tag{7.3}$$

with arbitrary, general speaking, functions $\varphi_n = \varphi_n(\mathbf{k})$; all terms of (4.6) apart the first three, as will be evident below, can be omitted.

Let's consider now covariant "quasilocal terms," FT of which can be added to (7.3). In the most general form they can be represented as

$$R_{qL} = a_0\delta(x^2) + a_1 n_\mu \partial_\mu \delta(x^2) + a_2 \partial_\mu^2 \delta(x^2) + \cdots , \tag{7.4}$$

n_μ is an arbitrary time-like vector.

At limiting by the first term of (7.4), needed for the Green functions of free fields, and as in the $(t, \mathbf{k})$-representation $\widehat{\mathbf{F}}_{\mathbf{k}}[\delta(x^2)] = (1/2\pi^2 k)\sin(k|t|)$, the argument of δ-function of (6.4) would be replaced on $[(1 + a_0 k^2)\tan(kt) - kt]$. So, the transcendental equation, roots of which determine properties of nonlocal functions, is slightly varied, but the existence of the main zero at $t = 0$ is not changed. Other quasilocal terms can be considered analogously and therefore the main conclusion about instantaneous transferring of excitations by tunneling does not change. Note here only possibilities of consideration of quasilocal terms in decomposition of projectors and accordingly in decomposition of response functions. These problems require further investigations in correlation with experimental data.

Let's consider some physical effects that can be described by the representation (7.3).

The first of them is the examination of possibility of non-local phenomena, i.e. a signal transferring with the faster-then-c speed. The substitution of (7.3) into the general relation (4.1) shows that such interaction can be transferred only instantaneously::

$$F_{NL}(t, \mathbf{r}) = \int d\mathbf{r}' \{\varphi_0(\mathbf{r} - \mathbf{r}') - \varphi_1(\mathbf{r} - \mathbf{r}')\partial/\partial t + \frac{1}{2}\varphi_2(\mathbf{r} - \mathbf{r}')\partial^2/\partial t^2\} I(t, \mathbf{r}'). \tag{7.5}$$

The instant transfer of these three characteristics is necessary and sufficient for the complete reestablishment of initial form of signal and its dynamics. Notice that such transferring does not contradict the common causality.

The peculiarities of such transition can be revealed by consideration of the energy-momentum representation. The substitution

$$(t, \mathbf{r}, s) \rightleftarrows (t, \mathbf{p}, m) \tag{7.6}$$

leads to the projector $\theta(E^2 - p^2 - m^2)$ and gives possibility to consider transitions which go beyond the E-$\mathbf{p}$-hyperboloid, i.e. the tunneling phenomena. In these variables, with $m = 0$ for brevity, (7.3) is rewritten for photons tunneling as

$$R_{\text{tunnel}}(E|\mathbf{r}) = A(P(E|\mathbf{r})) = \phi_0(\mathbf{r})\delta(E) + \phi_1(\mathbf{r})\delta'(E) + \phi_2(\mathbf{r})\delta''(E) + \cdots . \tag{7.7a}$$

It shows that tunnel transition goes without alteration of energy of photon passing under barrier. The inverse FT shows that

$$R_{\text{tunnel}}(t|\mathbf{r}) = \phi_0(\mathbf{r}) + t\ \phi_1(\mathbf{r}) + t^2\ \phi_2(\mathbf{r}) + \cdots . \tag{7.7b}$$

i.e. it allows, at $t = 0$, possibility of the instantaneous tunneling described by the function $\phi_0(\mathbf{r})$.

For more precise definition of functions that allow the tunneling, it seems convenient to transfer from the variable E to the limiting momentum $p_0 = (2mE)^{1/2}$. For this case in the expression (6.4) and (7.2) are needed the substitutions $\mathbf{k} \to \mathbf{r}$, $t \to p_0$ that leads to the response function (7.7) with $E \to p_0$. Hence it proves that at the described conditions tunneling process goes without change of photons parameters (we do not consider spins in this paper).

For completion of this consideration an estimation of tunneling distance is needed. For this purpose we consider the projector of jump over distance not less a at the x-axis: it must include light cones separated onto a distance a at the moment $t = 0$:

$$P_a(t, x) = \theta(t^2 - x^2) + \theta(t^2 - (x + a)^2). \tag{7.8}$$

Functions described jumping on distances $L \geq a$ satisfy the condition of orthogonality:

$$(1 - P_a(t, x))\, f_a(t, \mathbf{r}) = 0. \tag{7.9a}$$

The FT of projector in (7.9) over time at $x = 0$ is

$$\widehat{\mathbf{F}}_\omega[1 - P_a(t, 0)] = (1/2\pi i\omega)(1 + e^{i\omega a}). \tag{7.9b}$$

Hence the corresponding response function is represented as:

$$f_a(\omega, \mathbf{r}) = \psi(\omega, \mathbf{r})\, \omega\, \delta(1 + e^{i\omega a}). \tag{7.10}$$

The zeros of δ-function's argument are located at $\omega a = \pi(2n - 1)$, $n = 1, 2, \ldots$ and correspondingly distances of instantaneous jumps are proportional to halfwave lengths: $a = \frac{1}{2}\lambda(2n - 1)$, i.e. they exceed, even at $n = 1$, the value of uncertainty ($\Delta x \Delta k \geq \frac{1}{2}$,) and therefore must be measurable.

If considered process is determined by difference with energy of stable (or resonant) state $\Delta\omega = \omega - \omega_0$, the operation of subtraction becomes needed:

$$f_a(\omega|\mathbf{r}) - f_a(\omega_0|\mathbf{r}) \to f_a(\Delta\omega|\mathbf{r}). \tag{7.11}$$

With this substituon we receive that the distance of instantaneous tunneling is equal

$$a = (\pi/\Delta\omega)(2n - 1), \quad n = 1, 2, 3, \ldots, \tag{7.12a}$$

which just corresponds to experimental data (Chiao and Steinberg, 1998; Milonni, 2002; Nimtz, 1997; Nimtz and Heitman, 1997; Recami, 2001; Steinberg, 2002) at $n = 1$.

Thus these considerations prove

Theorem 10. *Superluminal transfer of excitations (jumps) through a linear passive substance can be affected by nothing but by the instantaneous tunneling of virtual particles; the tunneling distance is of order of a half wavelength corresponding to the deficiency in the energy relative to the nearest stable (resonance) state.*

Hence the condition of the theorem can be expressed in a form similar to the uncertainty principle:

$$a\Delta\omega = \pi \tag{7.12b}$$

and really such expression is proven also via the uncertainty considerations by the use of projectors features (Perel'man, submitted).

It is proved in addition that the nonlocality of electromagnetic field must be described by the 4-potential A_μ, but the fields **E** and **B** remain unconnected within the near field.

Notice also such formal possibility of interpretation of "superluminal" transitions. By the analogy with (6.5a) the projector of external part of E-$\mathbf{p}$-hyperboloid can be represented via the propagators of Klein-Gordon equation, e.g.

$$P_{\omega<k}(\omega, \mathbf{k}) = \theta(-k^2) = \int_0^\infty d\mu^2 \delta(\omega^2 - k^2 + \mu^2) = 2\pi \int_0^\infty d\mu^2 \Delta_1(\omega, \mathbf{k}, i\mu), \tag{7.13}$$

but with an imaginary mass. It leads to DRs of (6.5b) type for such "superluminal" particles:

$$f_{\text{tach}}(x) = 2\pi \int_0^\infty d\mu^2 \int d^4y \Delta_1(x - y, i\mu)\, f_{\text{tach}}(y). \tag{7.14}$$

Hence it describes the "superluminal" transfer by introduction of tachions with an imaginary mass (e.g. Feinberg, 1967; Recami and Mignani, 1974 and references therein.). In the sharp distinctions with our consideration such description allows nonlocal interaction at any distance, which contradicts all known experimental data and must be thetefore omitted.

Let's consider for completeness of picture the condition of orthogonality $P_{NL}(t|\mathbf{r})\, f_L(t|\mathbf{r}) = 0$.

The $(\omega|\mathbf{r})$-representation of the general projector of nonlocality

$$P_{NL}(\omega|\mathbf{r}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, \theta(R^2 - t^2) e^{-i\omega t} = \delta(\omega) - \sin(\omega R)/\pi\omega, \tag{7.15}$$

where $R^2 = \mathbf{r}^2 + s^2$. It leads at $\omega \neq 0$ to the representation of local functions:

$$f_L(\omega|\mathbf{r}) = \phi(\omega|\mathbf{r})\,\delta(\sin(\omega R)/\pi\omega) = \pi\phi(\omega|\mathbf{r})\sum_1^\infty \delta(\omega - \pi n/R). \qquad (7.16)$$

This representation shows that the contact interaction with $R \to 0$ of merging particles is possible at $\omega \to \infty$ only, hence confluent particles must be reduced into another state, into another particle.

In classical theory at fixed distance R between emitter and receiver (7.16) shows that they regularly interact by standing waves only. In quantum theory the impossibility of exact fixation of positions would softening such resonance of interactions.

8. NONRELATIVISTIC PROCESSES: SCHRÖDINGER EQUATION

Let's consider the introduction of such artificial "property." All scattering processes on a fixed point-like force center of arbitrary nature can be divided onto two kinematically isolated groups: events or processes, in the course of which total energy always bigger than kinetic energy, and such events or processes, at which kinetic energy can become bigger the total one (they include captures, reflections or backward scattering; tunneling).

We shall begin with a more simple dividing. If the total energy is precisely divided on kinetic and potential parts, i.e. $E = p^2/2m + V$ and, as a modelling example, the potential V is constant, the pointed equation of restrictions can be presented:

$$\psi(E, \mathbf{p}) = q(E, \mathbf{p})\,\delta(E - p^2/2m - V). \qquad (8.1)$$

with arbitrary function $q(E, \mathbf{p})$ nonsingular at poles of δ-function. Its FT,

$$\psi(t, \mathbf{r}) = \int dE d\mathbf{p}\, q(E, \mathbf{p})\,\delta(E - p^2/2m - V) \exp i(\mathbf{pr} - Et), \qquad (8.2a)$$

satisfies the differential equation

$$(i\partial_t - \nabla^2/2m - V)\,\psi(t, \mathbf{r}) = 0, \qquad (8.2b)$$

which with specification of potential $V \to V(\mathbf{r}, \ldots)$ directly leads to the Schrödinger equation.

Now we turn to more general cases. If the total energy of single scattering particle is bigger than kinetic energy during the process, the state is described by the equation of restriction:

$$\psi(t, \mathbf{r}) = \theta(E - p^2/2m)\,\psi(t, \mathbf{r}). \qquad (8.3)$$

The FT of this projector over all (3 + 1) variables

$$P_E(t, \mathbf{r}) = \frac{1}{2}\delta(t)\delta(\mathbf{r}) - m^{3/2}(2\pi t)^{-5/2}\exp(imr^2/2t + i\pi/4) \tag{8.4}$$

and can be rewritten via the Green function (propagator) of the Schrödinger equation,

$$g(t, \mathbf{r}) = 2\pi\alpha_0 \left(\frac{m}{i\hbar t}\right)^{3/2} \exp\left(\frac{imr^2}{2\hbar t}\right), \tag{8.5}$$

as

$$P_1(t, \mathbf{r}) = \frac{1}{2}\left(\delta(t)\delta(\mathbf{r}) - \frac{1}{2\pi t}g(t, \mathbf{r})e^{i\pi/4}\right) \equiv \frac{1}{2}[\delta(x) - \alpha_0 K(x)], \tag{8.6}$$

where $x = (t, \mathbf{r})$ and the complete numerical parameter is singled out:

$$\alpha_0 = 2(2\pi)^{-5/2} = 1/49.48\cdots \approx 1/50. \tag{8.7}$$

As must be underlined, the numerical value of (8.7) evidently depends on the number d of space dimensions only: $\alpha_0(d) = 1/\pi(2\pi)^{d/2}$; the exponent in (8.5) contains the elementary action function in the $\hbar$ units.

The processes with surplus energy non-passing through tunneling are described by response functions satisfying the DRs

$$f_E(t, \mathbf{r}) = -\alpha_0 m^{3/2} e^{i\pi/4} \int d\tau d\overrightarrow{\rho}\ f_E(t - \tau, \mathbf{r} - \overrightarrow{\rho})\, \tau^{-5/2}\exp(im\rho^2/2\tau). \tag{8.8}$$

The kernel of (8.8) has evident singularity at $\tau \to 0$ characterizing lesser probability of temporal delays in processes of such energies. More compactly these DRs can be rewritten via nondimensional magnitudes with the Compton wavelength $\lambda_C = \hbar/mc$: relative rapidity $u = \lambda_C/c\tau$ and relative length $\kappa = \rho/\lambda_C$ as

$$f_E(t, \mathbf{r}) = \alpha_0 \int du d\kappa\ f_E(t - \lambda_C/uc, \mathbf{r} - \lambda_C\kappa)\, u^{1/2}\exp\left(\frac{1}{2}i\kappa^2 u\right). \tag{8.9}$$

It seems very perspective, although it is very venturesome, to *suggest* that exactly this α_0 can be examined as *a candidate on the role of the universal bare constant of SUSY*. Preliminary in the favour of such hypothesis can be advanced such reasons: this numerical value is the consequence of conservation laws only, i.e. of kinematics, it appears beyond specification of interaction types and depends only on the space dimensions; in the construction of multiple DRs, the complicated matrix elements of (3.30)-type describing more than one interactions should be proportional to corresponding degrees of α_0. But with or without such conjectures all perspectives must be examined by a direct comparison with the existing field theory and by estimations of basic magnitudes.

Let's begin with a "naive" estimation of charges renormalization "constants" (more correctly, numerical values of running functions of renormalization at definite energies) via the numerical value of α_0. For renormalization of strong charge to m_Z energy with the experimental value $\alpha_s(m_Z) = 0.12$ (Hagiwara, 2002) it gives

$$Z_s^2(m_Z) = \alpha_0/\alpha_s(m_Z) \approx 49.48/8.33 \sim 6. \tag{8.10a}$$

For renormalization of electric charge to atomic magnitude it leads to an estimation:

$$Z_e^2(m_e) = \alpha_0/\alpha = 49.48/137 \approx 0.361 \sim 2^{-3/2}, \tag{8.10b}$$

and for weak interaction, correspondingly (Weinberg, 1998); θ_W is the running Weinberg angle),

$$Z_w^2 = \alpha_0/\alpha_w = \alpha_0/\alpha(m_Z)\cos^2\theta_W \sim 0.29. \tag{8.10c}$$

More consecutively α_0 and one from the known experimental values: $\alpha_s(m_Z)$, $\alpha_e(m_Z)$ or $\sin^2\theta_W$ can be taken as the input to estimate other parameters via the generalized Gell-Mann - Low relations (Gell-Mann and Low, 1954):

$$1/\alpha_0 - 1/\alpha_k = (\beta_k/2\pi)\ln(M/m_k), \tag{8.11}$$

where α_k are the running couplings, $k = e, s, w$; for the case of electromagnetic interactions $\alpha_e = \alpha(m_Z)\sin^2\theta_W(m_Z)$; at weak interactions $\alpha_w = (3/5)\alpha(m_Z)\cos^2\theta_W(m_Z)$ and $1/\alpha_0$ must be factorized on 3/5. In the one-loop approximation with omitting contributions from scalar bosons and so on (e.g. Weinberg, 2000) with suggestion of only three lepton families $\{\beta_s; \beta_e; \beta_w\} = \{7; 10/3; -4\}$. (Note that although β-functions are calculated till 8-th order (Nigam, 1999), for our qualitative estimations the lowest order seems sufficient.)

With the most accurately measured $1/\alpha_e(m_Z) = 129$ and $\sin^2\theta_W = 0.231$ it leads to $M_0 = 10^{18}$ GeV that is only twice bigger the value defined in Binger and Brodsky. But, as must be noted, for the proton lifetime, $\tau_p \sim M_0^4/\alpha_0^2 m_p^5$, these values lead to a non-contradict estimation: $\tau_p \sim 3.5 \cdot 10^{33}$ yrs.

The most problematic in (8.11) are values of beta-functions that depend on inclusion of different types of Higgs particles and symmetries partners into theory. For partial analysis of such possibilities can be used relations that follow the system (8.11):

$$\beta_e(1/\alpha_0 - 1/\alpha_s) = \beta_s(1/\alpha_0 - 1/\alpha_e); \tag{8.12}$$

$$\beta_w(1/\alpha_0 - 1/\alpha_s) = \beta_s(3/5\alpha_0 - 1/\alpha_w). \tag{8.13}$$

From the expression (8.12) with $\beta_s = 7$ and with the experimental values $\alpha_e' = \alpha_e(m_Z)\sin^2\theta_W = 1/29.8$ and $\alpha_s = 0.12$ follows $\beta_e = 3.34$ in the excellent agreement with the one-loop approximation, without inclusion of additional particles.

But the relation (8.13) at such conditions results in $\beta_w = -5.1$, markedly different from the one-loop value -4. The corresponding value of ratio $(\beta_w/\beta_e) = -3/2$ indicates also on the restricted electroweak unification or on the dynamical breaking of electroweak symmetry. (Such discrepancy can be connected, in principle, with our initial restriction (8.3), regarding, generally speaking, stable systems only.)

The including of all supersymmetry partners of quarks, leptons, gauge and Higgs bosons (Peskin) results in the system $\{\beta_s; \beta_e; \beta_w\}' = \{3, -1, -33/5\}$. But such values lead to too unreasonable magnitude M_0' of order 10^{39} GeV. This discrepancy can indicate on significant excess of frequently considered supersymmetry partners (there are other possibilities for estimation of beta-functions, see e.g. Terning).

More scrupulous calculations of beta-functions, with taking into account many-loops corrections and/or different variants of SUSY, will shift values of all beta-functions and can lead to refinements of mass spectra of MSSM. Note that the including of scalar particles and so on in all scheme (8.11) (Weinberg, 2000) can essentially decrease corresponding values and leads to discrepancy between involved magnitudes. Here it can be assumed that an existence of these additional particles, Higgs bosons and others, must be more carefully examined in the weak interaction relating to bigger masses of intermediate bosons.

The calculated value of M_0 is very close to the Planck mass $M_{Pl} = (8\pi G_N)^{-1/2} = 2.4 \cdot 10^{18}$ GeV with the Newtonian constant G_N and the non-dimensional gravitation constant $G_N M_0^2/\hbar c \approx 1/149 \approx \alpha_0/3$. It possibly indicates that at the unification limit $G_N \to G_0 \approx 3G_N$, which is of order of electromagnetic and weak interactions renormalization (cf., e.g. Binetruy). This renormalization must be taken into account as possibility of some changes of Planck units in $\sqrt{3}$ fold, especially at estimations of black holes characteristics. Note that the value (8.13) is deduced without any assumptions of SUSY, but it can be considered as an argument in favor of some unification of supergravity type.

Let's consider now possibilities of evaluating of masses of some stable particles via the magnitudes of α_0 and M_0.

The change (generation) of mass Δm_k by vacuum polarization is expressed as

$$\Delta m_k/m_k = -(\alpha_0 b_k/2\pi)\ln(M_0/m_k), \tag{8.14}$$

where the mass coefficients in the minimal SU(5) theory (Georgi and Glashow, 1974) are determined in the one-loop approximation (Cheng and Li, 1984) as $\{b_s;\ b_e;\ b_w\} = \{-8;\ -9/2;\ -9/10\}$.

The right side of (8.14) is equal to 1.07 for strong interactions with the nucleon mass. It allows to assert that the all observed mass of nucleon is of the pure field-born source. For electromagnetic interactions with electron mass the right side gives 0.71 that is of the specific order. The evident discrepancy can be connected with restricted calculations over the lowest order.

All these results require further examination, but they can point a perspective way for continuation of this approach.

9. CONCLUSIONS

The carried out considerations are such. On the one hand their purposes consisted in developing the general theory, which would allow the formal, i.e. in the mathematically strict form, description of some qualitative properties of physical systems. On the other hand there are some concrete problems of theory that must be investigated in the maximally general representation.

Let us briefly enumerate these considerations and their results.

1. Alternative properties of physical systems characterized by restricted supports of response functions are expressed in the form of equations of restriction, which by integral transformations lead to convolution equations, considered as spectral representations and dispersion relations. It allows their establishment beyond analyses of analyticity. This approach is naturally applied to functions of many variables with more complicated set of supports.

   The offered approach can be described as the chain of consecutive isomorphic transitions: property → support of parameters → projector → equation of restriction → spectral representation.
2. The general possibility of DRs in weak topology corresponding to symmetries of systems is mentioned.
3. The partial FT of equations of restriction over different set of variables can reveal various features of response functions.
4. The interrelations between DRs and equations of motion are established in the general form: strict detachment of admissible parameters → projector of this domain, usually of lower dimensions → equation of motion → Green function → integration over domain corresponding to the description of property.
5. On this base, as the examples of general theory, the different establishments of Kramers-Kronig DRs, including the relation for indices of refrac-

tion, and sampling theorems are considered. The systems of nonrelativistic many-particle DRs with their approximate forms are written out.

6. The relativistic one-particle DRs are established and some their features are considered (relativistic many particles DR are established on this base in Perel'man, 1969). Such generalization of the Kramers-Kronig relations corresponds to reciprocal Klein-Gordon equation describing duration and space extent of interaction. These DRs can be used for description of the space dispersion also.
7. The analyses of these DRs demonstrate the possibility of instantaneous transferring of excitations in the strict limits that are close, but slightly bigger the uncertainty principles values and therefore are measurable.
8. The nonrelativistic DRs for one-particle scattering on the fixed point-like center are established by pure kinematical considerations, independent from the type of interaction. This DR is characterized by the single numerical factor and it seems very desirable to consider it onto the role of the universal bare charge of SUSY and so on. The estimations of charges renormalization and masses variability on its base encourages to such possibilities, i.e. demonstrates perspectives of such approach, on the base of kinematics as the most general theory.
9. The comparison of (6.3a) and (6.3b) factually proves the Poincare-Bertrand theorem regulating permutation of integration order in singular integrals. It seems evident that the suggested method with functions of many variables leads to several generalizations of this theorem.

In the whole we can conclude that the discussion of properties of physical systems on the base of projectors represent a peculiar terra incognita or almost incognita. It requires and deserves more complete and careful further researches.

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